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Entropy Production per Site in (Nonreversible) Spin-Flip Processes

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Entropy production per site in a (nonreversible) spin-flip process is studied. We give it a useful expression, from which a property stronger than affinity of the entropy production per site follows. Furthermore, quasi-invariance of nonequilibrium measures in the spin-flip processes is discussed via entropy production.

KEY WORDS: Entropy production; spin-flip processes; nonreversible stationary measures; nonequilibrium measures.

1. INTRODUCTION

In 1971 Holley[7] proved a version of the well-known H-theorem of Boltzmann for infinite spin systems called stochastic Ising models, which are spin-flip processes with reversible Gibbs measures. After this result, similar ones were obtained for various stochastic reversible systems by many authors.[6,9,4] Common features among them are the following. The system under consideration is reversible with respect to Gibbs measures for some potential. The specific free energy associated with the potential defines the H-function. That is, letting $F(\cdot)$ be the specific free energy functional and μ_t be the distribution of the process at time t, one can show that $F(\mu_t)$ is nonincreasing in t.

This kind of result was extended by Künsch[8] to nonreversible systems with stationary measures having certain regularity properties. As will be seen later, all the results also can be restated as nonpositivity of the time derivative of the relative entropy with respect to the stationary measure. The quantity which we call entropy production is the one

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obtained from the derivative by changing only its sign. The main purpose of this paper is to find an identification for the thermodynamic limit of the entropy production. It will turn out that the functional obtained as the limit measures, in some sense, how the state is far from (or close to) equilibrium of the system.

We discuss a spin system on $\mathbf{Z}^d$, the d-dimensional integer lattice. Set $E = \{-1, 1\}^{\mathbf{Z}^d}$, whose elements are denoted by $\eta = (\eta_k; k \in \mathbf{Z}^d)$. We will use the notations $E_\Lambda = \{-1, 1\}^\Lambda$ and $\mathscr{F}_\Lambda = \sigma(\eta_k;\ k \in \Lambda)$ for $\Lambda \subset \mathbf{Z}^d$. The spin-flip process we consider is the Markov process on E with the (pre-) generator of the form

$$\mathscr{L}f(\eta) = \sum_{k \in \mathbf{Z}^d} c(\tau_k \eta)\, \nabla_k f(\eta), \qquad \nabla_k f(\eta) = f(\gamma_k \eta) - f(\eta) \tag{1.1}$$

where τ_k is the shift by k, i.e., $(\tau_k \eta)_i = n_{i+k}$, and $\gamma_k \eta$ denotes the configuration whose spins coincide with η except at k, at which site the spin is $-\eta_k$. The function $c \circ \tau_k$ is simply denoted by c_k. We assume that the jump rate c is a strictly positive continuous function such that

$$\left(\sum_{k \in \mathbf{Z}^d} \|\nabla_0 c_k\|_\infty = \right) \sum_{k \in \mathbf{Z}^d} \|\nabla_k c\|_\infty < \infty \tag{1.2}$$

Our argument will be crucially based on the existence of a stationary measure with the following properties due to Künsch.[8] Let ν be a stationary measure of the process associated with $\mathscr{L}$. Throughout this paper, we suppoe that its local conditional distributions defined as

$$p^k(\eta) = \nu(\omega_k = \eta_k \mid \mathscr{F}_{\{k\}^c})(\eta) \tag{1.3}$$

are strictly positive continuous functions such that $p^k(\eta) = p_0(\tau_k \eta)$ (shift invariance) and

$$\sum_k \|\nabla_k p^0\|_\infty < \infty \tag{1.4}$$

Under these conditions it is shown in ref. 8 that p^k are actually determined by the equation

$$\sum_k \nabla_0 (c_k - \hat{c}_k) = 0 \qquad \text{with} \quad \hat{c}_k(\eta) = c_k(\gamma_k \eta)\, p^k(\gamma_k \eta)/p^k(\eta) \tag{1.5}$$

We shall use the notation $\hat{c}$ instead of $\hat{c}_0$ so that $\hat{c}_k = \hat{c} \circ \tau_k$. It should be noted that the stationary measure ν is trivially a Gibbs measure for the local specification $\{p^V\}$ given by

$$p^V(\eta) = \nu(\omega_k = \eta_k, \forall k \in V \mid \mathscr{F}_{V^c})(\eta) \tag{1.6}$$

where V are finite sets in $\mathbf{Z}^d$. Each p^V is also a strictly positive continuous function on E. See ref. 10 for a detailed account of the local specifications and of the associated Gibbs measures.

We now define two functionals of probability measures on E, relative entropy and entropy production. For two given probability measures μ and λ on a measurable space, the relative entropy of μ with respect to λ is defined by

$$h(\mu, \lambda) = \begin{cases} E^\mu[\log(d\mu/d\lambda)] & \text{if } \mu \ll \lambda \\ \infty & \text{otherwise} \end{cases} \tag{1.7}$$

where $E^\mu[\cdot]$ denotes the expectation with respect to μ. Let $\mathscr{M}(E)$ be the totality of Borel probability measures on E and set

$$\mathscr{M}_0(E) = \{\mu \in \mathscr{M}(E); \mu \text{ is shift invariant}\}$$

Here $\mu \in \mathscr{M}(E)$ is said to be shift invariant if $\mu = \mu \circ \tau_k^{-1}$ for all $k \in \mathbf{Z}^d$. Set

$$h_\Lambda(\mu, \nu) := h(\mu|_{\mathscr{F}_\Lambda}, \nu|_{\mathscr{F}_\Lambda}) = E^\mu\left[\log \frac{d\mu}{d\nu}\bigg|_{\mathscr{F}_\Lambda}\right] \tag{1.8}$$

for $\mu \in \mathscr{M}(E)$ and $\Lambda \subset \mathbf{Z}^d$, where $d\mu/d\nu|_{\mathscr{F}_\Lambda}$ denotes the density of $\mu|_{\mathscr{F}_\Lambda}$ with respect to $\nu|_{\mathscr{F}_\Lambda}$. In the case when p^k are of the Gibbsian form

$$p^k(\gamma_k \eta)/p^k(\eta) = \exp\left(2 \sum_{V \ni k} J_V \prod_{i \in V} \eta_i\right) \tag{1.9}$$

with a shift invariant potential $\{J_V\}$ satisfying $\sum_{V \ni 0} |J_V| < \infty$, the free energy $F_\Lambda(\mu)$ (discussed in ref. 7) in Λ is shown to be related to $h_\Lambda(\mu, \nu)$ in the following fashion:

$$F_\Lambda(\mu) = h_\Lambda(\mu, \nu) - P_\Lambda + o(|\Lambda|) \qquad \text{as} \quad \Lambda \uparrow \mathbf{Z}^d \tag{1.10}$$

where P_Λ are constants for which $\lim |\Lambda|^{-1} P_\Lambda$ (called the pressure of the potential $\{J_V\}$) exists. In (1.10) and in what follows, $\Lambda \uparrow \mathbf{Z}^d$ means that Λ runs over hypercubes with center 0 and $|\Lambda|$ denotes the cardinality of Λ. The relation (1.10) implies that, up to the smaller order term in the volume, the 'time derivative' of the free energy coincides with that of the relative entropy. The meaning of 'time derivative' is made clear in the next definition of *entropy production*. Given $\mu \in \mathscr{M}(E)$ and bounded $\Lambda \subset \mathbf{Z}^d$, we define entropy production in Λ by

$$\sigma_\Lambda(\mu) := -\frac{d}{dt} h_\Lambda(\mu_t, \nu)|_{t=0} = E^\mu\left[(-\mathscr{L}) \log \frac{d\mu}{d\nu}\bigg|_{\mathscr{F}_\Lambda}\right] \tag{1.11}$$

where μ_t is the distribution of the process at time t with the initial measure μ. See Lemma 2.2 below for a detailed account and for the proof of the last equality, where the convention $0 \log 0 = 0$ is implicitly used.

By setting

$$\sigma(\mu) = \liminf_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \sigma_\Lambda(\mu) \tag{1.12}$$

some of the main results obtained by Künsch,[8] which are extensions of Holley's,[7] are described as follows.

Theorem.[8] If the local conditional distributions of the Gibbsian form (1.9) satisfy Eq. (1.5) and $\sum_k \|\nabla_k \hat{c}\|_\infty < \infty$, then:

1. $\sigma(\mu)$ is nonnegative for all $\mu \in \mathscr{M}(E)$.
2. For $\mu \in \mathscr{M}_\theta(E)$, liminf in (1.12) can be actually replaced by lim.
3. On $\mathscr{M}_\theta(E)$, $\sigma(\mu)$ is lower semicontinuous in μ, and takes value 0 if and only if μ is a Gibbs measure for $\{J_V\}$.

We note that the Gibbsian specification (1.9) satisfies both (1.4) and $\sum_k \|\nabla_k \hat{c}\|_\infty < \infty$ if $\sum_{V \ni 0} |V| \cdot |J_V| < \infty$.

In the rest of this paper we are only concerned with measures μ in $\mathscr{M}_\theta(E)$. We will show the existence of $\lim |\Lambda|^{-1} \sigma_\Lambda(\mu) =: \sigma(\mu)$ without Gibbsian form of the local conditional distributions. Further, the main purpose of this paper is to study $\sigma(\cdot)$ as a functional on $\mathscr{M}_\theta(E)$, and in this situation we call $\sigma(\mu)$ the *entropy production per site* of μ associated with the spin-flip process. We now state our main result.

Theorem 1. Let μ be in $\mathscr{M}_\theta(E)$. Then $\lim |\Lambda|^{-1} \sigma_\Lambda(\mu) =: \sigma(\mu)$ exists and

$$\sigma(\mu) = E^\mu[c]\, h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \tag{1.13}$$

where $\mu^\alpha \in \mathscr{M}(E)$ ($\alpha = c$ or $\hat{c}$) is given by $d\mu^\alpha = E^\mu[\alpha]^{-1}\, \alpha\, d\mu$. In addition, if $\sigma(\mu) < \infty$, then

$$\frac{1}{|\Lambda|}(-\mathscr{L}) \log \frac{d\mu}{d\nu}\bigg|_{\mathscr{F}_\Lambda} \to E^\mu\left[c(\cdot) \log \frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}} \,\bigg|\, \mathscr{I} \right] \tag{1.14}$$

in $L^1(\mu)$ as $\Lambda \uparrow \mathbf{Z}^d$, where $\mathscr{I}$ denotes the shift-invariant σ-field on E.

The convergence (1.14) can be regarded as the one behind the existence of the thermodynamic limit of $|\Lambda|^{-1} \sigma_\Lambda(\mu)$. We should note that a similar quantity to the right-hand side of (1.13) appeared in ref. 12 for a general reversible process, but was not identified with the associated entropy production per site. Our identity (1.13) is useful for deriving properties of $\sigma(\cdot)$ itself. For example, it gives the following, which immediately implies that σ is affine on $\mathscr{M}_\theta(E)$.

Corollary 2. Let $\mathscr{T}$ denote the tail σ-field on E and $\mathscr{I}$ be as in Theorem 1. Then there exists a $\mathscr{T} \cap \mathscr{I}$-measurable function σ^*: $E \to [0, \infty]$ such that

$$\sigma(\mu) = E^\mu[\sigma^*] \qquad \text{for all} \quad \mu \in \mathscr{M}_\theta(E) \tag{1.15}$$

Proofs of Theorem 1 and Corollary 2 are given in Section 2.

Next we state a quasi-invariance result obtained as another corollary to Theorem 1. To do this, let us introduce the notion of quasi-invariance. Given $\Lambda \subset \mathbf{Z}^d$, consider the mapping γ_Λ on E defined by

$$(\gamma_\Lambda \eta)_i = \begin{cases} -\eta_i & \text{if } \ i \in \Lambda \\ \eta_i & \text{if } \ i \notin \Lambda \end{cases} \tag{1.16}$$

This is called the modification in Λ. Let Γ denote the family of finite modifications; $\Gamma = \{\gamma_\Lambda;\ |\Lambda| < \infty\}$, or equivalently Γ is the group of mappings generated by $\{\gamma_i;\ i \in \mathbf{Z}^d\}$. We say that $\mu \in \mathscr{M}(E)$ is *Γ-quasi-invariant* if μ and $\mu \circ \gamma^{-1}$ are equivalent for all $\gamma \in \Gamma$. It is easy to see that any Gibbs measure for an arbitrary absolutely summable potential is Γ-quasi-invariant. The next result concerns this property in the nonequilibrium situation.

Corollary 3. Suppose that the initial measure μ of the spin-flip process is shift invariant and let μ_t be the distribution of the process at time t. Then:

(i) For a.a. $t > 0$, μ_t is Γ-quasi-invariant.

(ii) For all $t > 0$, the time-averaged measure $t^{-1} \int_0^t \mu_s\, ds$ is Γ-quasi-invariant.

We shall prove this and discuss Γ-quasi-invariance in more detail in Section 3. Note that in view of the results of Sullivan,[12] the quasi-invariance property of nonequilibrium measures like Corollary 3 could be seen for more general infinite systems. The associated quasi-invariance there would be determined by the dynamics. For instance, consider so-called stochastic lattice gases. These are particle systems in $E' := \{0, 1\}^{\mathbf{Z}^d}$ with conservation law for the 'particle number.' Under suitable assumptions, the quasi-invariance in this context is described in terms of the family Π of mappings on E' generated by $\{\pi_{ij};\ i, j \in \mathbf{Z}^d\}$, where π_{ij} is defined by

$$(\pi_{ij}\eta)_k = \begin{cases} \eta_j & \text{if } k = i \\ \eta_i & \text{if } k = j \\ \eta_k & \text{if } k \neq i \text{ and } k \neq j \end{cases}$$

In the reversible case, this assertion can be proved by similar arguments to those given in the following sections for our spin-flip process. Indeed, a version of Theorem 1 holds true as is stated in the following. Consider the Markov process in E' governed by the generator

$$\mathscr{G}f(\eta) = \sum_{ij} c(ij, \eta)\, \nabla_{ij} f(\eta), \qquad \nabla_{ij} f(\eta) = f(\pi_{ij}\eta) - f(\eta)$$

where ij stands for a two-point set $\{i, j\}$ in $\mathbf{Z}^d$. Suppose that the family $\{c(ij, \cdot)\}_{ij}$ satisfies the conditions (i)–(iii) of ref. 4, p. 78. We also suppose that $c(0i, \cdot) \equiv 0$ except for finite number of i's, for which $c(0i, \cdot)$ are strictly positive continuous functions. If the process associated with $\mathscr{G}$ is reversible with respect to a Gibbs measure for a shift-invariant potential $\{J_V\}$ with $\sum_{V \ni 0} |J_V| < \infty$, i.e., for each $ij \subset \mathbf{Z}^d$,

$$\begin{aligned} c(ij, \eta) \exp\Bigl(-\sum_{V \cap ij \neq \varnothing} J_V \prod_{k \in V} \eta_k\Bigr) \\ = c(ij, \pi_{ij}\eta) \exp\Bigl(-\sum_{V \cap ij \neq \varnothing} J_V \prod_{k \in V} (\pi_{ij}\eta)_k\Bigr) \end{aligned} \tag{1.17}$$

then the entropy production per site associated with this process exists for all $\mu \in \mathscr{M}_0(E')$ and is expressed as

$$\tfrac{1}{2} \sum_{i : c(0i, \cdot) > 0} E^{\mu}[c(0i, \cdot)]\, h(\mu^{c(0i, \cdot)}, \mu^{c(0i, \cdot)} \circ \pi_{0i}^{-1}) \tag{1.18}$$

The first half of the above assertion is essentially implied by Remark (3.41) of ref. 4.

We finally remark that, without the reversibility (1.17), Spohn[11] studied stationary measures of the process associated with $\mathscr{G}$. He used the entropy production method and obtained analogous results to Künsch's.[8]

2. PROOF OF THEOREM 1

In this section we shall give the proof of Theorem 1 after giving a series of lemmas. Throughout this section the following notations are used. Let $k \in \mathbf{Z}^d$. The configuration $\gamma_k \eta$ is simply denoted by η^k. The cylinder set determined by $\eta \in E_\Lambda$ or the restriction of $\eta \in E$ on a $\Lambda \subset \mathbf{Z}^d$ is written as $[\eta]$ or $[\eta]_\Lambda$, respectively. Thus

$$[\eta] = \{\omega \in E; \omega_k = \eta_k, k \in \Lambda\} \qquad \text{for} \quad \eta \in E_\Lambda$$

$$[\eta]_\Lambda = \{\omega \in E; \omega_k = \eta_k, k \in \Lambda\} \qquad \text{for} \quad \eta \in E$$

Our first task is to look for a uniform bound of the local specifications p^Λ.

Lemma 2.1. There exists a constant $C_1 \in (0, \infty)$ such that

$$e^{-C_1|\Lambda|} \leqslant \inf_{\eta \in E} p^\Lambda(\eta) \tag{2.1}$$

for all bounded $\Lambda \subset \mathbf{Z}^d$.

Proof. Fix a bounded $\Lambda \subset \mathbf{Z}^d$ and an $\eta \in E$. By the definition (1.6) of the local specification, for $k \in \Lambda$ and $\omega \in E$

$$\frac{p^\Lambda(\omega)}{p^\Lambda(\omega^k)} = \frac{p^k(\omega)}{p^k(\omega^k)} \tag{2.2}$$

Take an arbitrary $\omega \in [\eta]_{\Lambda^c}$. Then there exists $\{k_1,..., k_N\} \subset \Lambda$ with $N \leqslant |\Lambda|$ such that $\eta = \gamma_{k_1} \circ \cdots \circ \gamma_{k_N} \omega$. So

$$\begin{aligned} \left|\log \frac{p^\Lambda(\eta)}{p^\Lambda(\omega)}\right| &\leqslant \sum_{j=1}^{N} \sup_{\xi \in [\eta]_{\Lambda^c}} |\log p^{k_j}(\xi) - \log p^{k_j}(\xi^{k_j})| \\ &\leqslant N \, \|\nabla_0 \log p^0\|_\infty \leqslant |\Lambda| \cdot \|\nabla_0 \log p^0\|_\infty \end{aligned}$$

This implies the inequality

$$p^\Lambda(\eta) \geqslant p^\Lambda(\omega) \exp(-|\Lambda| \cdot \|\nabla_0 \log p^0\|_\infty)$$

Summing up both sides over $\omega \in [\eta]_{\Lambda^c}$, we have

$$2^{|\Lambda|} p^\Lambda(\eta) \geqslant \exp(-|\Lambda| \cdot \|\nabla_0 \log p^0\|_\infty)$$

This proves that (2.1) is valid with $C_1 = \log 2 + \|\nabla_0 \log p^0\|_\infty$. ∎

By Lemma 2.1, we see that $\nu([\eta]) > 0$ for all $\eta \in E_\Lambda$ and bounded $\Lambda \subset \mathbf{Z}^d$. So, given $\mu \in \mathcal{M}(E)$, the local density ϕ_Λ of $\mu|_{\mathcal{F}_\Lambda}$ with respect to $\nu|_{\mathcal{F}_\Lambda}$ is defined for all bounded $\Lambda \subset \mathbf{Z}^d$. We can regard it as not only a function of $\eta \in E$, but also a function of $\eta \in E_\Lambda$. In terms of this density, a more explicit expression for the entropy production σ_Λ than (1.11) is given as follows.

Lemma 2.2. For a $\mu \in \mathscr{M}(E)$ and a bounded $\Lambda \subset \mathbf{Z}^d$,

$$\sigma_\Lambda(\mu) = \sum_{k \in \Lambda} \sum_{\eta \in E_\Lambda} \int_{[\eta]} c_k \, d\mu \cdot \{\log \phi_\Lambda(\eta) - \log \phi_\Lambda(\eta^k)\}$$
$$= E^\mu[(-\mathscr{L}) \log \phi_\Lambda] \tag{2.3}$$

where the convention $0 \log 0 = 0$ is used, so that the right-hand side of (2.3) takes value ∞ if and only if both $\phi_\Lambda(\eta) > 0$ and $\phi_\Lambda(\eta^k) = 0$ hold for some $k \in \Lambda$ and $\eta \in E_\Lambda$.

Proof. The proof is essentially the same as that given in ref. 4, and involves standard calculations based on the forward equation of the process associated with $\mathscr{L}$. We omit it. ▌

As in Theorem 1, we use the notation $d\mu^\alpha := E^\mu[\alpha]^{-1} \alpha \, d\mu$ ($\alpha = c$ or $\hat{c}$). Note that these two normalizing constants are equal if μ is shift invariant. Indeed, we have the following result.

Lemma 2.3. If $\mu \in \mathscr{M}_0(E)$, then $E^\mu[c] = E^\mu[\hat{c}]$.

This lemma is a consequence of the relation (1.5). Although it has been essentially proved in ref. 8 (Theorem 4.1), we shall give the proof for the reader's convenience.

Proof of Lemma 2.3. Let $\Lambda \subset \mathbf{Z}^d$ be bounded and $\eta \in E$ be fixed. Set $\varphi_\Lambda = \sum_{k \in \Lambda} (c_k - \hat{c}_k)$. Using the definition (1.5) of $\hat{c}_k$ and (2.2), one can easily show that

$$\sum_{\omega \in [\eta]_{\Lambda^c}} \varphi_\Lambda(\omega) \, p^\Lambda(\omega) = 0 \tag{2.4}$$

so that

$$\inf_{\omega \in [\eta]_{\Lambda^c}} \varphi_\Lambda(\omega) \leqslant 0 \leqslant \sup_{\omega \in [\eta]_{\Lambda^c}} \varphi_\Lambda(\omega) \tag{2.5}$$

Let ω' and $\omega'' \in [\eta]_{\Lambda^c}$ satisfy

$$\sup_{\omega \in [\eta]_{\Lambda^c}} \varphi_\Lambda(\omega) = \varphi_\Lambda(\omega'), \qquad \inf_{\omega \in [\eta]_{\Lambda^c}} \varphi_\Lambda(\omega) = \varphi_\Lambda(\omega'') \tag{2.6}$$

respectively. Then

$$\sup_{\omega \in [\eta]_{\Lambda^c}} |\varphi_\Lambda(\omega)| \leqslant \varphi_\Lambda(\omega') - \varphi_\Lambda(\omega'') \leqslant \sum_{k \in \Lambda} \|\nabla_k \varphi_\Lambda\|_\infty$$
$$\leqslant \sum_{k \in \Lambda} \sum_{i \notin \Lambda} (\|\nabla_k c_i\|_\infty + \|\nabla_k \hat{c}_i\|_\infty)$$

Here, the first inequality and the third one follow from (2.5) and (1.5), respectively, and the rightmost side is of $o(|\Lambda|)$ as $\Lambda \uparrow \mathbf{Z}^d$ by the assumption (1.2) and $\sum_k \|\nabla_k \hat{c}\|_\infty < \infty$, which is proved by showing

$$\|\nabla_k \hat{c}\|_\infty \leqslant \|\nabla_k c\|_\infty \left\| \frac{p^0 \circ \gamma_0}{p^0} \right\|_\infty + 2 \|c\|_\infty \|p^0\|_\infty \|\nabla_k p^0\|_\infty \left\| \frac{1}{p^0} \right\|_\infty^2$$

and then using (1.2) and (1.4).

Now, use the shift invariance of μ to show

$$\begin{aligned} &|\Lambda| \times |E^\mu[c] - E^\mu[\hat{c}]| \\ &\quad = |E^\mu[\varphi_\Lambda]| \leqslant \sup_{\omega \in E} |\varphi_\Lambda(\omega) - \varphi_\Lambda(\omega \cdot \eta)| + \sup_{\omega \in [\eta]_{\Lambda^c}} |\varphi_\Lambda(\omega)| \end{aligned}$$

where $\omega \cdot \eta$ is defined as

$$(\omega \cdot \eta)_i = \begin{cases} \omega_i & \text{if } \ i \in \Lambda \\ \eta_i & \text{if } \ i \in \Lambda^c \end{cases}$$

It remains to show that the first term in the rightmost side is also of $o(|\Lambda|)$. But this is a consequence of the continuity of $c - \hat{c}$. In fact, it is easy to prove that for all continuous functions f on E

$$\sum_{k \in \Lambda} \sup_{\omega \in E} |f_k(\omega) - f_k(\omega \cdot \eta)| = o(|\Lambda|)$$

as $\Lambda \uparrow \mathbf{Z}^d$, where $f_k = f \circ \tau_k$. ∎

The next lemma shows that the generalization of the Shannon–McMillan theorem can be applied to find the thermodynamic limit of the integrand in (1.11). Given $\Lambda \subset \mathbf{Z}^d$ and $k \in \mathbf{Z}^d$, let $\Lambda - k = \{i - k;\, i \in \Lambda\}$.

Lemma 2.4. Let μ be in $\mathscr{M}_0(E)$ and $\Lambda \subset \mathbf{Z}^d$ be bounded. Suppose that $\sigma_\Lambda(\mu) < \infty$. Then μ-almost surely it holds that

$$(-\mathscr{L}) \log \phi_\Lambda(\eta) = \sum_{k \in \Lambda} c(\tau_k \eta) \log \left. \frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}} \right|_{\mathscr{F}_{\Lambda - k}} (\tau_k \eta) + R_\Lambda(\eta) \qquad (2.7)$$

with $R_\Lambda(\eta) = o(|\Lambda|)$ uniformly in $\eta \in E$ and $\mu \in \mathscr{M}_0(E)$ as $\Lambda \uparrow \mathbf{Z}^d$.

Proof. For notational simplicity, set

$$\alpha_{k,\Lambda}(\eta) = \begin{cases} \displaystyle\int_{[\eta]_\Lambda} \alpha_k \, d\mu & \text{for } \ \eta \in E \\ \displaystyle\int_{[\eta]} \alpha_k \, d\mu & \text{for } \ \eta \in E_\Lambda \end{cases} \qquad (\alpha = c \text{ or } \hat{c})$$

Fix an $\eta \in E$ such that $\mu([\eta]_\Lambda) > 0$. Observe from Lemma 2.3 that

$$\left.\frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}}\right|_{\mathscr{F}_{\Lambda-k}} (\tau_k \eta) = c_{k,\Lambda}(\eta)/\hat{c}_{k,\Lambda}(\eta^k) \tag{2.8}$$

which is well defined for all $k \in \Lambda$ by the assumption $\sigma_\Lambda(\mu) < \infty$ and Lemma 2.2. Put

$$\psi_\Lambda = \left.\frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}}\right|_{\mathscr{F}_V}$$

when it is well defined for a $V \subset \mathbf{Z}^d$. Using (2.8) and $c_k(\eta)\, p^k(\eta) = \hat{c}_k(\eta^k)\, p^k(\eta^k)$, we obtain

$$\begin{aligned}
R_\Lambda(\eta) &= (-\mathscr{L}) \log \phi_\Lambda(\eta) - \sum_{k \in \Lambda} c(\tau_k \eta) \log \psi_{\Lambda-k}(\tau_k \eta) \\
&= \sum_{k \in \Lambda} c(\tau_k \eta) \left\{ \log \frac{\mu([\eta]_\Lambda)}{c_{k,\Lambda}(\eta)} - \log \frac{\mu([\eta^k]_\Lambda)}{\hat{c}_{k,\Lambda}(\eta^k)} - \log \frac{\nu([\eta]_\Lambda)}{\nu([\eta^k]_\Lambda)} \right\} \\
&= \sum_{k \in \Lambda} c(\tau_k \eta) \left\{ \log \frac{c_k(\eta)\, \mu([\eta]_\Lambda)}{c_{k,\Lambda}(\eta)} - \log \frac{\hat{c}_k(\eta^k)\, \mu([\eta^k]_\Lambda)}{\hat{c}_{k,\Lambda}(\eta^k)} \right. \\
&\qquad \left. - \log \frac{\nu([\eta]_\Lambda)\, p^k(\eta^k)}{\nu([\eta^k]_\Lambda)\, p^k(\eta)} \right\}
\end{aligned}$$

The following bounds are easy to show:

$$\left| \log \frac{c_k(\eta)\, \mu([\eta]_\Lambda)}{c_{k,\Lambda}(\eta)} \right| \leqslant \sup_{\omega \in [\eta]_\Lambda} |\log c_k(\omega) - \log c_k(\eta)|$$

$$\left| \log \frac{\hat{c}_k(\eta^k)\, \mu([\eta^k]_\Lambda)}{\hat{c}_{k,\Lambda}(\eta^k)} \right| \leqslant \sup_{\omega \in [\eta^k]_\Lambda} |\log \hat{c}_k(\omega) - \log \hat{c}_k(\eta^k)|$$

On the other hand, by (2.2),

$$\nu([\eta]_\Lambda) = \int \nu(d\omega)\, p^\Lambda(\eta \cdot \omega) = \int \nu(d\omega)\, p^\Lambda(\eta^k \cdot \omega)\, \frac{p^k(\eta \cdot \omega)}{p^k(\eta^k \cdot \omega)}$$

where $\eta \cdot \omega \in E$ is given by

$$(\eta \cdot \omega)_i = \begin{cases} \eta_i & \text{if } i \in \Lambda \\ \omega_i & \text{if } i \in \Lambda^c \end{cases}$$

Hence one can get

$$\left|\log \frac{\nu([\eta]_\Lambda)\, p^k(\eta^k)}{\nu([\eta^k]_\Lambda)\, p^k(\eta)}\right| \leqslant \sup_{\omega \in [\eta]_\Lambda} |\log p^k(\omega) - \log p^k(\eta)|$$
$$+ \sup_{\omega \in [\eta^k]_\Lambda} |\log p^k(\omega) - \log p^k(\eta^k)|$$

The above three estimates and the continuity of $\log c$ and $\log p^0$ together complete the proof of Lemma 2.4, since the following is true for an arbitrary continuous function f on E:

$$\sum_{k \in \Lambda} \sup\{|f_k(\omega) - f_k(\eta)|;\, \omega, \eta \in E,\, \omega|_\Lambda = \eta|_\Lambda\} = o(|\Lambda|)$$

as $\Lambda \uparrow \mathbf{Z}^d$, where $f_k = f \circ \tau_k$. ∎

Denote by $\tilde{\sigma}_\Lambda(\mu)$ the μ-expectation of the summation in the right-hand side of (2.7) if $\sigma_\Lambda(\mu) < \infty$. Otherwise set $\tilde{\sigma}_\Lambda(\mu) = \infty$. It is important to note that $\tilde{\sigma}_\Lambda(\mu)$ essentially contributes to the existence of the thermodynamic limit of $|\Lambda|^{-1}\, \sigma_\Lambda(\mu)$ as follows.

Lemma 2.5. Fix a $\mu \in \mathscr{M}_0(E)$. Then $\Lambda \mapsto \tilde{\sigma}_\Lambda(\mu) \in [0, \infty]$ is superadditive, namely

$$\tilde{\sigma}_{\Lambda_1}(\mu) + \tilde{\sigma}_{\Lambda_2}(\mu) \leqslant \tilde{\sigma}_{\Lambda_1 \cup \Lambda_2}(\mu) \tag{2.9}$$

holds for bounded Λ_1, $\Lambda_2 \subset \mathbf{Z}^d$ such that $\Lambda_1 \cap \Lambda_2 = \varnothing$. In particular, $\sigma(\mu) = \lim |\Lambda|^{-1}\, \sigma_\Lambda(\mu) \in [0, \infty]$ exists and

$$\sigma(\mu) = \lim_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|}\, \tilde{\sigma}_\Lambda(\mu) = \sup_{\Lambda;\text{cube}} \frac{1}{|\Lambda|}\, \tilde{\sigma}_\Lambda(\mu) \tag{2.10}$$

Proof. From (2.7) and $E^\mu[c] = E^\mu[\hat{c}]$, it is not difficult to observe that for all bounded $\Lambda \subset \mathbf{Z}^d$,

$$\tilde{\sigma}_\Lambda(\mu) = E^\mu[c] \sum_{k \in \Lambda} h_{\Lambda - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \geqslant 0 \tag{2.11}$$

This implies that

$$\begin{aligned}\tilde{\sigma}_{\Lambda_1 \cup \Lambda_2}(\mu) &= E^\mu[c] \sum_{k \in \Lambda_1} h_{\Lambda - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) + E^\mu[c] \sum_{k \in \Lambda_2} h_{\Lambda - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \\ &\geqslant E^\mu[c] \sum_{k \in \Lambda_1} h_{\Lambda_1 - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) + E^\mu[c] \sum_{k \in \Lambda_2} h_{\Lambda_2 - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \\ &= \tilde{\sigma}_{\Lambda_1}(\mu) + \tilde{\sigma}_{\Lambda_2}(\mu)\end{aligned}$$

whenever $\Lambda_1 \cap \Lambda_2 = \varnothing$. With the help of Lemma 2.4, the second assertion is proved by standard argument. [See, for instance, ref. 5, Lemma (15.11).] ∎

We now state a generalization of Perez's theorem given by Fritz[3] in a special form that is suitable for our purpose.

Lemma 2.6 (Perez, Fritz). Let $\mu_i \in \mathcal{M}(E)$ $(i=1,2)$ and let $\mathcal{A}$ be the family of bounded sets of $\mathbf{Z}^d$. If

$$\sup_{V \in \mathcal{A}} h_V(\mu_1, \mu_2) < \infty \tag{2.12}$$

then $\mu_1 \ll \mu_2$,

$$h(\mu_1, \mu_2) = \sup_{V \in \mathcal{A}} h_V(\mu_1, \mu_2) \tag{2.13}$$

and

$$\log \frac{d\mu_1}{d\mu_2}\bigg|_{\mathcal{F}_V} \to \log \frac{d\mu_1}{d\mu_2} \qquad \text{in} \quad L^1(\mu_1) \qquad \text{as} \quad V \to \mathbf{Z}^d$$

in the sense that for each $\varepsilon > 0$ there exists a $V_\varepsilon \in \mathcal{A}$ such that

$$E^{\mu_1} \left| \log \frac{d\mu_1}{d\mu_2}\bigg|_{\mathcal{F}_V} - \log \frac{d\mu_1}{d\mu_2} \right| < \varepsilon \qquad \text{whenever} \quad V \supset V_\varepsilon$$

As an application of this lemma we now prove Theorem 1.

Proof of Theorem 1. Suppose that $\mu \in \mathcal{M}_0(E)$ is given. It is obvious by (2.11) that

$$\tilde{\sigma}_\Lambda(\mu) \leqslant |\Lambda| \; E^\mu[c] \; h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1})$$

and so by Lemma 2.5

$$\sigma(\mu) \leqslant E^\mu[c] \; h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \tag{2.14}$$

which implies that $h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) = \infty$ whenever $\sigma(\mu) = \infty$. In the rest of the proof we assume that $\sigma(\mu) < \infty$. We need to verify that the condition (2.12) holds with $\mu_1 = \mu^c$, $\mu_2 = \mu^{\hat{c}} \circ \gamma_0^{-1}$. For each $V \in \mathcal{A}$,

$$\begin{aligned} h_V(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) &= \left\{ \lim_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \sum_{k \in \Lambda : \Lambda - k \supset V} 1 \right\} h_V(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \\ &\leqslant \limsup_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} h_{\Lambda - k}(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \\ &= E^\mu[c]^{-1} \lim_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \tilde{\sigma}_\Lambda(\mu) = E^\mu[c]^{-1} \, \sigma(\mu) \end{aligned}$$

and hence

$$\sup_{V \in \mathscr{A}} h_V(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) \leqslant E^\mu[c]^{-1} \sigma(\mu) < \infty$$

Combining this with (2.13) and (2.14), we obtain the first assertion of Theorem 1,

$$\sigma(\mu) = E^\mu[c] \, h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1})$$

Set, as in the proof of Lemma 2.4,

$$\psi_\Lambda = \left. \frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}} \right|_{\mathscr{F}_\Lambda} \qquad \text{and} \qquad \psi = \frac{d\mu^c}{d\mu^{\hat{c}} \circ \gamma_0^{-1}}$$

Each of the logarithms of these functions is in $L^1(\mu^c)$ by a general inequality in ref. 1:

$$\int \left| \log \frac{d\mu_1}{d\mu_2} \right| d\mu_1 \leqslant h(\mu_1, \mu_2) + [2h(\mu_1, \mu_2)]^{1/2} \tag{2.15}$$

which holds for arbitrary probability measures μ_i $(i = 1, 2)$ such that $\mu_1 \ll \mu_2$ on a measurable space. Moreover, the above inequality together with $h(\mu^c, \mu^{\hat{c}} \circ \gamma_0^{-1}) < \infty$ implies

$$C_2 := \max\left\{ \sup_{\Lambda \in \mathscr{A}} E^{\mu^c} |\log \psi_\Lambda|, E^{\mu^c} |\log \psi| \right\} < \infty$$

Now by Lemma 2.6, for each $\varepsilon > 0$ we can find a $V_\varepsilon \in \mathscr{A}$ such that

$$E^{\mu^c} |\log \psi_\Lambda - \log \psi| \leqslant \varepsilon \qquad \text{for all} \quad \Lambda \supset V_\varepsilon$$

and hence it holds that

$$\begin{aligned} &\frac{1}{|\Lambda|} \sum_{k \in \Lambda} E^{\mu^c} |\log \psi_{\Lambda - k} - \log \psi| \\ &\qquad \leqslant \varepsilon + \frac{2C_2}{|\Lambda|} |\{k \in \Lambda; \Lambda - k \text{ does not contain } V_\varepsilon\}| \to \varepsilon \end{aligned}$$

as $\Lambda \uparrow \mathbf{Z}^d$. Consequently,

$$\lim_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \sum_{k \in \Lambda} E^{\mu^c} |\log \psi_{\Lambda - k} - \log \psi| = 0$$

Combining this with the mean ergodic theorem, we have

$$\frac{1}{|\Lambda|}\sum_{k\in\Lambda} c(\tau_k\eta)\log\psi_{\Lambda-k}(\tau_k\eta)\to E^{\mu}[c\log\psi\mid\mathscr{I}]$$

in $L^1(\mu)$ as $\Lambda\uparrow\mathbf{Z}^d$, where $\mathscr{I}$ is the shift-invariant σ-field. Therefore the proof of the second assertion of Theorem 1 is completed by Lemma 2.4. ∎

Proof of Corollary 2. Denote by $\mathscr{F}$ the Borel σ-field of E. Let $\mathscr{T}$ be the tail σ-field on E. It is known [see Theorem (14.10) in ref. 5] that we can construct a version λ^{ω} of the conditional distribution given $\mathscr{I}$ satisfying the following:

1. For all $\omega\in E$, $\lambda^{\omega}(\cdot)\in\mathscr{M}_0(E)$
2. For all $A\in\mathscr{F}$, $\omega\mapsto\lambda^{\omega}(A)$ is $\mathscr{I}\cap\mathscr{T}$-measurable.
3. For all $\mu\in\mathscr{M}_0(E)$ and $A\in\mathscr{F}$

$$\lambda^{\omega}(A)=\mu(A\mid\mathscr{I})(\omega)\qquad\mu\text{-a.a. }\omega$$

Define $\sigma^*\colon E\to[0,\infty]$ by $\sigma^*(\omega)=\sigma(\lambda^{\omega})$. It follows from (2.10) and condition 2 of the above proof that σ^* is $\mathscr{I}\cap\mathscr{T}$-measurable. The required identity $\sigma(\mu)=E^{\mu}[\sigma^*]$ is now proved by an almost similar argument to that in the proof of Theorem (15.20) in ref. 5. ∎

3. QUASI-INVARIANCE OF NONEQUILIBRIUM MEASURES

Γ-quasi-invariance introduced in Section 1 comes from the dynamics of the spin-flip process. This property, however also has been discussed in equilibrium statistical mechanics[10]; a Gibbs measure $\mu\in\mathscr{M}(E)$ for the local specification $\{p^V\}$ is characterized as a Γ-quasi-invariant measure such that

$$\frac{d(\mu\circ\gamma_k^{-1})}{d\mu}(\eta)=\frac{p^k(\gamma_k\eta)}{p^k(\eta)}\qquad\text{for all}\quad k\in\mathbf{Z}^d \tag{3.1}$$

Before proving Corollary 3 stated in the Introduction, we give some conditions equivalent to Γ-quasi-invariance.

Lemma 3.1. For given $\mu\in\mathscr{M}(E)$, the following three conditions are equivalent.

(i) μ is Γ-quasi-invariant.

(ii) μ and $\mu\circ\gamma_k^{-1}$ are equivalent for all $k\in\mathbf{Z}^d$.

(iii) For all bounded $\Lambda \subset \mathbf{Z}^d$ and all $\eta \in E_\Lambda$,

$$\mu([\eta] \mid \mathscr{F}_{\Lambda^c}) > 0 \qquad \mu\text{-a.s.} \tag{3.2}$$

Proof. Since every finite modification is represented as a composition of a finite number of one-point modifications, the proof of equivalence among above three conditions is elementary and omitted. ▮

The essential idea of the proof of Corollary 3 is that any shift-invariant measure with finite entropy production per site is Γ-quasi-invariant. Indeed, take $\mu \in \mathscr{M}_\theta(E)$ and suppose that $\sigma(\mu) < \infty$. Then, by (1.13), $\mu \ll \mu \circ \gamma_0^{-1}$, which also implies $\mu \circ \gamma_0^{-1} \ll \mu$. So μ and $\mu \circ \gamma_0^{-1}$ are equivalent, and the condition (ii) of Lemma 3.1 is verified by shift invariance of μ.

Proof of Corollary 3. Fix a $\mu \in \mathscr{M}_\theta(E)$ and denote by μ_t the distribution of the spin-flip process at time t with the initial measure μ. By a shift invariance of our dynamics [see (1.1)] μ_t is easily shown to be shift invariant. The Markov property and the definition (1.11) of $\sigma_\Lambda(\cdot)$ together yield

$$\frac{d}{dt} h_\Lambda(\mu_t, \nu) = -\sigma_\Lambda(\mu_t) \tag{3.3}$$

or

$$h_\Lambda(\mu, \nu) - h_\Lambda(\mu_t, \nu) = \int_0^t \sigma_\Lambda(\mu_s)\, ds \tag{3.4}$$

for all $t > 0$ and bounded $\Lambda \subset \mathbf{Z}^d$. Here it should be noted that $\sigma_\Lambda(\mu_t)$ is finite for all $t > 0$. In fact, if we assume that $\mu_t([\eta]) = 0$ for some $\eta \in E_\Lambda$, then

$$0 = \frac{d}{dt} \mu_t([\eta]) = \sum_{k \in \Lambda} \left\{ \int_{[\eta^k]} c_k\, d\mu_t - \int_{[\eta]} c_k\, d\mu_t \right\}$$

and these two relations imply that $\mu_t([\eta^k]) = 0$ for all $k \in \Lambda$. With the help of Lemma 2.2, we have $\sigma_\Lambda(\mu_t) < \infty$.

By Lemma 2.4 the identity (3.4) can be rewritten in the form

$$h_\Lambda(\mu, \nu) - h_\Lambda(\mu_t, \nu) = \int_0^t \tilde{\sigma}_\Lambda(\mu_s)\, ds + o(|\Lambda|) \tag{3.5}$$

Here, it is easy to observe from Lemma 2.1 that the left-hand side is dominated by $C_1\, |\Lambda|$. We now take $\Lambda = \Lambda_n$ $(n = 1, 2,\ldots)$, where

$$\Lambda_n = \{(k_1,\ldots, k_d) \in \mathbf{Z}^d;\ 1 \leqslant k_i \leqslant 2^n,\ i = 1,\ldots, d\}$$

Using Lemma 2.5 and the shift invariance of μ, one can see without difficulty that $|\Lambda_n|^{-1} \tilde{\sigma}_{\Lambda_n}(\mu)$ is nondecreasing in n. So letting $\Lambda = \Lambda_n$ in (3.5) and then dividing both sides of (3.5) by $|\Lambda_n|$, we see from Lemma 2.5 and the monotone convergence theorem that

$$\int_0^t \sigma(\mu_s)\, ds \leqslant C_1 < \infty, \qquad t > 0 \tag{3.6}$$

We conclude that $\sigma(\mu_t) < \infty$ for a.a. $t > 0$. This together with the observation given before this proof implies that μ_t is Γ-quasi-invariant for a.a. $t > 0$, namely the assertion (i) of Corollary 3.

The second assertion is similary proved by only noting

$$\int_0^t \sigma(\mu_s)\, ds = t \cdot \sigma\left(\frac{1}{t} \int_0^t \mu_s\, ds\right)$$

which is an immediate consequence of Corollary 2. ∎

We finally explain how the entropy production per site describe the 'distance' between a given measure and equilibrium. Take a $\mu \in \mathscr{M}_0(E)$. As was shown after the proof of Lemma 3.1, $\sigma(\mu) = \infty$ whenever μ is not Γ-quasi-invariant. Next consider the case when μ is Γ-quasi-invariant. Put

$$U_0 = \log \frac{d\mu^c}{d(\mu^{\hat{c}} \circ \gamma_0^{-1})}$$

Then by Theorem 1 we have $\sigma(\mu) = E^\mu[cU_0]$, and by using (1.5) one can verify

$$\frac{d(\mu \circ \gamma_0^{-1})}{d\mu}(\eta) = \frac{p^0(\gamma_0 \eta)}{p^0(\eta)} e^{-U_0} \tag{3.7}$$

Comparing (3.7) with (3.1), we can regard the term U_0 as something like a perturbation from the equilibrium, and $\sigma(\mu)$ is thought of as a measure of this part.

Remark. Recently, Dai Pra[2] studied a functional $I(\cdot)$ on $\mathscr{M}_0(E)$ which is obtained as the rate function of the 'space-time' large deviations for a (nonreversible) spin-flip process. It was shown there that the functional vanishes exactly on the set of stationary measures of the process. It would be of interest to find an identification of $I(\cdot)$ to express the difference between $I(\cdot)$ and $\sigma(\cdot)$ in an explicit way. A very formal argument using the results in ref. 2 yields a conjecture that $I(\mu)$ is equal, up to some multiplicative constant, to the right side of (1.13) with h replaced by the square of the so-called Hellinger distance. But so far we have not succeeded in proving the conjecture.

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